LOCALLY WEAKLY FLAT SPACES(1)

BY
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1. **Introduction.** Let M be a topological m-manifold embedded in the Euclidean n-space E^n. A general problem is to devise local conditions on the embedding of M that imply some niceness or smoothness condition globally. An example in the case $n=m+1$, and, M a closed manifold, is the result of M. Brown asserting that if M is locally bicollared, then M is bicollared.

Local properties generally may be subdivided into two mutually exclusive categories, the medial variety(2) and the strictly local. Examples of the first kind are local compactness, local flatness, etc. Examples of the second kind are local connectedness, local weakly flatness, etc.

The present paper contains results showing that under the codimension restriction ($k \neq n-2$) certain strictly local properties about an embedding imply medial properties.

In the special case M is closed, $n=m+1$, and M is locally weakly flat (LWF, see definition below), it was shown in [10], that the set of points at which M fails to be locally flat cannot have any isolated points.

Recently it has been shown that an embedding of a k-manifold in E^n cannot fail to be locally flat at an isolated interior point, when $n-k \neq 2$ and $n \geqq 4$, or at an isolated boundary point when $n \geqq 4$ and $k \leqq n$. Using these results we show that a k-manifold M in E^n is LF if it is LWF at all points, $n \geqq 4$, and $n-k \neq 2$. Moreover, for the case $n \geqq 4$ and $k=n-2$ it is shown that M is LF if M is LF at each interior point and LWF at each boundary point. This is particularly interesting in the case $n-4$ since M can fail to be LF at an arc of the boundary of M which is LF in both E^4 and in the boundary of M. However, as a corollary to the latter result M cannot fail to be locally flat at the points of a Cantor set $C \subset \text{Bd } M$ which is tame in Bd M and in E^4.

Theorem 3.3 below shows that if a Cantor set in E^n has a certain local property, then the Cantor set is tame. This proves very useful in showing that the LWF property in codimension not equal to two implies tameness.

There are examples in codimension 2 for $n \geqq 3$ of manifolds having isolated wild points at which the manifold is LWF.

2. **Notations and definitions.** Let E^n denote n-dimensional Euclidean space and consider E^k as being embedded in the natural way in E^n whenever $n \geqq k$. Denote

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(2) This terminology is due to R. L. Wilder.

by $B^n(p, r)$ the ball in E^n of radius r and center at $p \in E^n$ and by E^n_+ the closed Euclidean half space determined by the inequality $x_n \geqq 0$. By manifold we mean topological manifold with boundary. If M is a manifold, denote by int M and bd M the interior and boundary of M. If B is an n-cell and C is an m-cell such that $C \subset B$ and $C \cap \text{int } B = \text{int } C$, then we say that (B, C) is a cell pair of type (n, m). The pair (B, C) is nice on its boundary if bd C is locally flat in bd B and there exists a homeomorphism $H: (\text{bd } B, \text{bd } C) \times [0, 1] \to (B, C)$ such that $H(x, 0) = x$ for all $x \in \text{bd } B$. Let M be an m-manifold in the interior of an n-manifold N. Recall that M is *locally flat* at $p \in \text{int } M$ if there exists a neighborhood B of p in N such that $(B, B \cap M)$ is a trivial cell pair of type (n, m); that is, homeomorphic to $(B^n(0, 1), B^m(0, 1))$. We say that M is *locally weakly flat* at $x \in \text{int } M$ if for each $\varepsilon > 0$ there is a neighborhood B of x in N such that diam $B < \varepsilon$ and $(B, B \cap M)$ is a cell pair of type (n, m) which is nice on its boundary. The notion of locally weakly flat at a boundary point will be considered in §5.

From this point on we assume that M is an m-manifold, N is an n-manifold, $p \in \text{int } M$, $M \subset \text{int } N$, and M is locally weakly flat at each point of some neighborhood of p. Let B be an n-cell neighborhood of p such that $(B, B \cap M)$ is a cell pair of type (n, m) which is nice on its boundary and M is locally weakly flat at each point of int $B \cap M$. Let Y denote the singular set of $B \cap M$. (The singular set of X is the set of points at which X fails to be locally flat.)

3. **The structure of the singular set.** We first state the following unknotting theorem for locally flat cell pairs. This result is proved in [8].

LEMMA 3.1. *Let (B, C) be a cell pair of type (n, m) which is nice on its boundary and suppose that C is locally flat except possibly at one interior point. If $n > 3$ and $n - m \neq 2$ then (B, C) is trivial.*

THEOREM 3.2. *If $n > 3$ and $n - m \neq 2$ then Y is a Cantor set or the empty set.*

Proof. Y is clearly closed. Since M is locally weakly flat at each point of int $B \cap M$, Y is totally disconnected. If $y \in Y$ is an isolated point, then there exists a neighborhood B_y in int B such that $(B_y, B_y \cap M)$ is a cell pair of type (n, m) which is nice on its boundary and such that $B_y \cap Y = y$. By Lemma 3.1, M is locally flat at y. Hence Y has no isolated point

We eventually will show that Y is empty when $n - m \geqq 3$ or when $n \geqq 4$ and $n - m = 1$. First we show that Y is tame.

THEOREM 3.3. *Let C be a Cantor set in E^n and suppose that for each point $p \in C$ and each $\varepsilon > 0$ there exists an n-cell B such that $p \in \text{int } B$, $\text{bd } B \cap C = \varnothing$, and* diam $B < \varepsilon$. *Then C is tame.*

Proof. This theorem generalizes a theorem of Bing [1, Corollary 3.2], however, the proof of Bing's theorem uses 3-dimensional surgery and does not easily generalize. Osborne [16] has shown that if for each $\varepsilon > 0$ there exists a finite number of pairwise disjoint n-cells $B_1, \ldots, B_k$ such that $C \subset \bigcup_{i=1}^{k} \operatorname{int} B_i$ and diam $B_i < \varepsilon$ then C is tame.

Let $\varepsilon > 0$ be given: we will construct the collection of pairwise disjoint n-cells as in Osborne's theorem. Let A be an arc such that $C \subset A \subset E^n$ and let $A_1, \ldots, A_k$ be disjoint subarcs of A such that $C \subset \bigcup_{i=1}^{k} A_i$ and diam $A_i < \varepsilon/3$ for $i = 1, 2, \ldots, k$. Let $\varepsilon' = \min \{d(A_i, A_j) \mid i \neq j\}$ and let $\delta = \min(\varepsilon/6, \varepsilon'/4)$. Cover $A_1 \cap C$ by n-cells $D_1, \ldots, D_{k_1}$ such that diam $D_i < \delta$, bd $D_i \cap C = \varnothing$, and $(\operatorname{int} D_t - \bigcup_{i=1}^{t-1} D_i) \cap C \neq \varnothing$ for $t = 2, 3, \ldots, k_1$. Let A_1' be a pwl δ' approximation of A_1 where $\delta' < \delta$ is so small that $A_1' \cap (\operatorname{int} D_t - \bigcup_{i=1}^{t-1} D_i) \neq \varnothing$ for $t = 2, 3, \ldots, k_1$. By Whitehead's regular neighborhood theorem [Corollary 1_n, p. 293 of [17]], there exists an n-ball B_1' such that $A_1' \subset \operatorname{int} B_1'$ and $B_1' \subset N(A_1', \delta) \subset N(A_1, 2\delta)$. We shall push B_1' onto $B_1 \subset N(A_1, 2\delta)$ so that $C \cap A_1 \subset \operatorname{int} B_1$.

Since $C \cap D_1$ is a compact subset of int D_1, there exists a homeomorphism f_1 of E^n such that $f_1 \mid E^n - D_1 = 1 \mid E^n - D_1$, and $f_1(\operatorname{int} B_1') \supset C \cap D_1$. Suppose that a homeomorphism f_t of E^n has been constructed so that

(I_t) $f_t \mid E^n - \bigcup_{i=1}^{t} D_i = 1 \mid E^n - \bigcup_{i=1}^{t} D_i$, and

(II_t) $f_t(\operatorname{int} B_1') \supset C \cap \bigcup_{i=1}^{t} D_i$.

Since $D_{t+1} \cap C$ is a compact subset of int D_{t+1} and $B_1' \cap (\operatorname{int} D_{t+1} - \bigcup_{i=1}^{t} D_i) \neq \varnothing$, by I_t there exists a homeomorphism f_{t+1}' of E^n such that

(III) $f_{t+1}' \mid E^n - D_{t+1} = 1 \mid E^n - D_{t+1}$, and

(IV) $f_{t+1}' f_t(\operatorname{int} B_1') \supset C \cap D_{t+1}$.

Let $f_{t+1} = f_{t+1}' f_t$. Then I_{t+1} follows immediately from I_t and III. Also, II_{t+1} follows from II_t, III, and IV. Let $B_1 = f_{k_1} B_1'$. $B_1 \subset N(A_1', \delta) \cup (\bigcup_{i=1}^{k_1} D_i)$. Therefore $B_1 \subset N(A_1, 2\delta) \subset N(A_1, \varepsilon'/2)$ and diam $B_1 < 2\delta + \varepsilon/3 + 2\delta \leqq \varepsilon$. Furthermore $B_1 \supset C \cap (\bigcup_{i=1}^{k_1} D_i) = C \cap A_1$. We may go through the same construction for each i obtaining n-cells $B_1, B_2, \ldots, B_k$ such that $A_i \cap C \subset \operatorname{int} B_i \subset N(A_i, \varepsilon'/2)$ and diam $B_i < \varepsilon$. Since $d(A_i, A_j) \geqq \varepsilon'$ if $i \neq j$, the n-cells $B_1, \ldots, B_k$ are pairwise disjoint and the required collection of n-cells has been constructed.

We next list a theorem of Bryant [3] to be used below.

Theorem 3.4 [Bryant]. *Suppose f is an embedding of a k-dimensional polyhedron X^k into a combinatorial n-manifold M where $2k+2 \leqq n$, and P is a tame polyhedron in M^n, with* dim $P \leqq n/2 - 1$, *such that $f \mid (X^k - f^{-1}(P))$ is locally tame. Then f is ε-tame.*

Theorem 3.5. *If $n > 3$ and $n - m \neq 2$ then Y lies on an arc A which is tame both in $M \cap$* int B *and in* int B.

Proof. Since M is locally weakly flat at each point of Y, for each $\varepsilon > 0$ the Cantor set Y can be covered by a finite number of m-cells $C_1, \ldots, C_t$ in $B \cap M$ such that bd $C_i \cap Y \neq \varnothing$ and diam $C_i < \varepsilon$ for $i = 1, 2, \ldots, t$. Therefore the hypotheses of Theorem 3.3 are satisfied and Y lies on an arc (in $B \cap M$) which is tame in int $B \cap M$. Therefore there exists a homeomorphism f: $[0, 1] \to \operatorname{int} B \cap M$ such that $A = f([0, 1]) \supset Y$ and A is tame in int $B \cap M$. Since M is locally flat except possibly at the points of Y, A is locally flat in int B except possibly at the points of Y. However, again by Theorem 3.3, Y lies on an arc $P\subset$ int B which is tame in int B. Since $f\mid([0,1]-f^{1}(P))$ is locally flat and P is tame in int B it follows from Theorem 3.4 that A is ε-tame and hence tame.

We now state for reference the following theorem.

THEOREM 3.6. *Suppose S is a k-sphere topologically embedded in E^n, $n\geqq 4$, $n-k\neq 2$. Then the singular set of S is either empty or uncountable.*

This theorem is a summary of results due to several authors: Chernavskii [7] for $n\geqq 5$, Kirby [13] for $n-k=1$, Cantrell and Lacher [6] for $n-k\geqq 3$, and Hutchinson [12] for $n\geqq 5$, $n-k=1$. In fact, most of the results quoted are somewhat stronger in special cases than Theorem 3.6 but we state only the result needed here.

THEOREM 3.7. *Suppose $n-m\neq 2$ and $n>3$. For each $\varepsilon>0$ there exists a finite number of n-cells $B_1,\ldots,B_k$ in* int B *such that the following conditions are satisfied:*

(1) $B\cap B_j=\varnothing$ *whenever* $i\neq j$,

(2) $(B_i, B_i\cap M)$ *is a cell pair of type* (n, m) *which is nice on its boundary* (*see* §2),

(3) $Y\subset\bigcup_{i=1}^{k}$ int B_i, *and*

(4) diam $B_i<\varepsilon$.

Proof. Let A be an arc as in Theorem 3.5. There exist subarcs $A_1, A_2,\ldots, A_k$ of A such that $Y\subset\bigcup_{i=1}^{k}A_i$, $A_i\cap A_j=\varnothing$ for $i\neq j$, and diam $A_i<\varepsilon/2$. Since A is tame in int B and in int $B\cap M$, A_i is cellular in both int B and int $B\cap M$. Let $f\colon B\to B$ be a map such that $f(A_i)=p_i$, a point, and $f\mid B-\bigcup A_i$ is a homeomorphism. Since A_i is cellular in $B\cap M$, $f(B\cap M)$ is homeomorphic to $B\cap M$ and since $f\mid B-\bigcup_{i=1}^{k}A_i$ is a homeomorphism, $f(B\cap M)$ is locally flat in B except possibly on p_i. Let

$$(S^{n'}, S^{m'}) = \text{cone}\,\{\text{bd}\, B, f(\text{bd}\, B\cap M)\}\cup(B, f(B\cap M)).$$

Since $(B, f(B\cap M))$ is nice on its boundary, $(S^{n'}, S^{m'})$ is a sphere pair of type (n, m) such that $S^{m'}$ is locally flat in $S^{n'}$ except possibly at p_i. By Theorem 3.6 $S^{m'}$ is locally flat. Therefore $f(B\cap M)$ is locally flat in B at p_i. Let B_1' be n-cells in int B such that $p_i\in$ int B_i', $f^{-1}(B_1')\subset N(A_i, \varepsilon/4)$, $B_i'\cap B_j'=\varnothing$ for $i\neq j$, and $(B_i', B_i'\cap f(M))$ is a trivial cell pair of type (n, m). Since $p_i\in$ int B_i', f^{-1} is a homeomorphism on a neighborhood of bd B_i'. Since $(B_i', B_i'\cap f(M))$ is nice on its boundary it follows from the Generalized Schoenflies Theorem [2] that

$$(f^{-1}(B_i'), f^{-1}(B_i'\cap f(M)))$$

is a cell pair of type (n, m). Let $B_i=f^{-1}(B_i')$.

We shall show that $B_1, B_2,\ldots, B_k$ satisfy the conclusion of the theorem. Since diam $A_i<\varepsilon/2$ and $B_i\subset N(A_i, \varepsilon/4)$, diam $B_i<\varepsilon$. Since $B_i'\cap B_j'=\varnothing$ for $i\neq j$, $B_i\cap B_j=\varnothing$ for $i\neq j$. Since $Y\subset\bigcup_{i=1}^{k}A_i\subset\bigcup_{i=1}^{k}$ int B_i, condition (3) is satisfied. It remains to show that $(B_i, B_i\cap M)$ is nice on its boundary. Let

$$G^i\colon(\text{bd}\, B_i', \text{bd}\, B_i'\cap f(M))\times[0,1]\to(B_i', B_i'\cap f(M))$$

be a homeomorphism such that $G^i(x, 0)=x$ for all $x \in \text{bd } B_i'$ and $G^i(\text{bd } B_i' \times [0, 1]) \cap \{p_i\} = \varnothing$. Define

$$H^i: (\text{bd } B_i, \text{bd } B_i \cap M) \times [0, 1] \to (B_i, B_i \cap M)$$

by $H^i(x, t)=f^{-1}(G^i(f(x), t))$. Since $p_i \notin G^i(\text{bd } B_i' \times [0, 1])$, H^i is a homeomorphism and $H^i(x, 0)=f^{-1}(G^i(f(x), 0))=f^{-1}(f(x))=x$ for all $x \in \text{bd } B_i$. Therefore $(B_i, B_i \cap M)$ is nice on its boundary. This completes the proof of Theorem 3.7. We next establish a condition on a cell pair which in codimension other than two implies the triviality of the cell pair.

4. A triviality condition.

THEOREM 4.1. *Let (B, D) be a cell pair of type (n, m) which is nice on its boundary and suppose $n>3$ and $n-m\neq 2$. Let Y denote the singular set of D. Then (B, D) is trivial if the following condition holds for each positive number ε. There exists a finite number of n-cells $B_1, \ldots, B_k$ in* int B *such that*:

1. $B_i \cap B_j = \varnothing$ *for* $i\neq j$,
2. $(B_i, B_i \cap D)$ *is a cell pair of type* (n, m) *which is nice on its boundary*,
3. $Y \subset \bigcup \text{int } B_i$, *and*
4. diam $B_i < \varepsilon$.

Proof. Suppose (B, D) is a cell pair of type (n, m) possessing the conditions of the theorem. Since $(\text{bd } B, \text{bd } D)$ is a locally flat sphere pair of codimension $\neq 2$, $(\text{bd } B, \text{bd } D)$ is trivial. Therefore there exists a homeomorphism

$$h_0: (\text{bd } B, \text{bd } D) \to (\text{bd } B^n(0, 1), \text{bd } B^m(0, 1)).$$

We shall extend h_0 to a homeomorphism $h: (B, D) \to (B^n(0, 1), B^m(0, 1))$. Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers converging to zero. Substituting ε_1 for ε and C_i' for B_i there exists a finite number s_1 of n-cells $C_i' \subset \text{int } B$ such that

1. $C_i' \cap C_j' = \varnothing$ for $i\neq j$,
2. $(C_i', C_i' \cap D)$ is a cell pair of type (n, m) which is nice on its boundary,
3. $Y \subset \bigcup \text{int } C_i'$, and
4. diam $C_i' < \varepsilon_1$.

Let $H^i: (\text{bd } C_i', \text{bd } C_i' \cap D) \times [0, 1] \to (C_i', C_i' \cap D)$ be the homeomorphism, guaranteed by 2, which satisfies $H^i(x, 0)=x$ for all $x \in \text{bd } C_i'$. Let (C_i'', D_i'') be the abstract cone over $(\text{bd } C_i', \text{bd } C_i' \cap D)$ with cone point c_i and with $(\text{bd } C_i', \text{bd } C_i' \cap D)$ identified with $(\text{bd } C_i', \text{bd } C_i' \cap D) \times 0$. Since $(\text{bd } C_i', \text{bd } C_i' \cap D)$ is a locally flat sphere pair (see 2, above) and $n-m\neq 2$, $(\text{bd } C_i', \text{bd } C_i' \cap D)$ is trivial and therefore (C_i'', D_i'') is trivial. Define $\pi_i: (C_i', C_i' \cap D) \to (C_i'', D_i'')$ by $\pi_i(H^i(x, t))=(x, t)$ for all $x \in \text{bd } C_i'$ and $0 \leqq t < 1$ and $\pi_i(C_i' - H^i(\text{bd } C_i' \times [0, 1)))=c_i$. Define (C_i'', D_i'') to be

$$\left((C-\bigcup C_i') \cup \bigcup_{i=1}^{s_1} C_i'', (D-\bigcup (C_i' \cap D)) \cup \bigcup_{i=1}^{s_1} D_i''\right)$$

and define $\pi: (B, D) \to (C'', D'')$ by $\pi \mid C_i' = \pi_i$ and $\pi \mid B-\bigcup_{i=1}^{s_1} C_i' = \text{identity} \mid B - \bigcup_{i=1}^{s_1} C_i'$. Clearly (C'', D'') is locally flat. Since $n-m\neq 2$ and the hypotheses of

Lemma 3.1 are fulfilled, (C'', D'') is trivial. Thus we may extend $h_0\pi^{-1} \mid \text{bd } C''$ to a homeomorphism $h_1'\colon (C'', D'') \to (B^n(0, 1), B^m(0, 1))$.

Let $\varepsilon_1' < \varepsilon_1$ be so small that $B^n(h(c_i), \varepsilon_1') \subset \text{int } h_1'(C_1'')$. Define (C_i, D_i) to be

$$\pi^{-1}h_1'^{-1}(B^n(h_1'(c_i), \varepsilon_1')), \qquad \pi^{-1}h_1'^{-1}(B^m(h_1'(c_i), \varepsilon_1'))$$

and let $h_1 = h_1'\pi \mid (B - \bigcup_{i=1}^{s_1} \text{int } C_i, D - \bigcup_{i=1}^{s_1} \text{int } D_i)$. Then the cell pairs (C_i, D_i) satisfy the following conditions:

(a) $C_i \subset \text{int } B$, $i = 1, 2, \ldots, s_1$,

(b) $C_i \cap C_j = \varnothing$, $i \neq j$,

(c) (C_i, D_i) is nice on its boundary,

(d) $Y \subset \bigcup_{i=1}^{s_1} \text{int } D_i$,

(e) diam $C_i < \varepsilon_1$ and diam $h_1(C_i) < \varepsilon_1$, $i = 1, 2, \ldots, s_1$,

(f)
$$h_1 \mid \left(B - \bigcup_{i=1}^{s_1} \text{int } C_i,\ D - \bigcup_{i=1}^{s_1} \text{int } D_i\right) \to \left(B^n(0, 1) - \bigcup \text{int } B^n(h(c_i), \varepsilon_1'),\ B^m(0, 1) - \bigcup_{i=1}^{s_1} \text{int } B^m(h_1(c_i), \varepsilon_k')\right)$$

is an onto homeomorphism, and

(g) $h_1 \mid (\text{bd } D, \text{bd } D) = h_0 \mid (\text{bd } B, \text{bd } D)$.

Inductively we can construct a sequence of homeomorphic extensions h_i and n-cells $C_{i_1 \ldots i_t}$ such that the following conditions are satisfied for each integer t.

(a_t) $C_{i_1 \ldots i_t} \subset \text{int } C_{i_1 \ldots i_{t-1}}$, for $1 \leqq i_t \leqq s_t$;

(b_t) $C_{i_1 \ldots i_t} \cap C_{j_1 \ldots j_t} = \varnothing$, unless $(i_1, \ldots, i_t) = (j_1, \ldots, j_t)$;

(c_t) $(C_{i_1 \ldots i_t}, D_{i_1 \ldots i_t})$ is a cell pair of type (n, m) which is nice on its boundary where $D_{i_1 \ldots i_t} = C_{i_1 \ldots i_t} \cap D$;

(d_t) $Y \subset \bigcup \{\text{int } D_{i_1 \ldots i_t} \mid 1 \leqq i_1 \leqq s_1, \ldots, 1 \leqq i_t \leqq s_t\}$;

(e_t) diam $C_{i_1 \ldots i_t} < \varepsilon_t$;

(f_t)
$$h_t\colon (B - \bigcup \text{int } C_{i_1 \ldots i_t}, D - \bigcup \text{int } D_{i_1 \ldots i_t}) \to ((B^n(0, 1) - \bigcup \text{int } B^n(p_{i_1 \ldots i_t}, \varepsilon_t'), B^m(0, 1) - \bigcup \text{int } B^m(p_{i_1 \ldots i_t}, \varepsilon_t')))$$

is an onto homeomorphism, where $p_{i_1 \ldots i_t} \in B^m(0, 1)$, $\varepsilon_t' < \varepsilon_t$, and

$$h_t(\text{bd } C_{i_1 \ldots i_t}, \text{bd } D_{i_1, \ldots i_t}) = (\text{bd } B^n(p_{i_1 \ldots i_t}, \varepsilon_t'), \text{bd } B^m(p_{\cdot 1 \ldots i_t}, \varepsilon_t'));$$

and

(g_t) $h_t \mid (B - \bigcup \text{int } C_{i_1 \ldots i_{t-1}}) = h_{t-1} \mid (B - \bigcup \text{int } C_{i_1 \ldots i_{t-}})_1$.

Now we shall define $h\colon (B, D) \to (B^n(0, 1), B^m(0, 1))$ which will be shown to be an onto homeomorphism. For $x \in B - Y$ let $h(x) = h_t(x)$ where t is some integer so large that $x \in B - \bigcup C_{i_1 \ldots i_t}$. If $x \in Y$ then there exists a sequence $i_1, i_2, \ldots$ such that $x = \bigcap_{t=1}^{\infty} \{D_{i_1 \ldots i_t}\}$ and a point $p = \bigcap_{t=1}^{\infty} \{B(p_{i_1 \ldots i_t}, \varepsilon_t')\}$. Define $h(x) = p$. Since $\lim \varepsilon_i = 0 = \lim \varepsilon_i'$ the continuity of h follows from the definition of h and conditions (f_t) and (g_t). In order to see that h is one-to-one let x_1 and x_2 be distinct points of B. There exists an integer t such that $\{x_1, x_2\} \not\subset C_{i_1 \ldots i_t}$ for any $(i_1, \ldots, i_t)$. Therefore $h(x_1) \neq h(x_2)$ follows from (f_t) and (g_t). Since $h: (B, D) \to (B^n(0, 1), B^m(0, 1))$ is one-to-one and continuous and extends

$$h_0: (\text{bd } B, \text{bd } D) \to (\text{bd } B^n(0, 1), \text{bd } B^m(0, 1)),$$

h is an onto homeomorphism. This completes the proof of Theorem 4.1. We may now apply Theorem 3.7 and Theorem 4.1 to the pair $(B, B \cap M)$ of §2.

THEOREM 4.2. *Let M and N be m- and n-manifolds respectively, $n > 3$ and $n - m \neq 2$, and suppose $M \subset \text{int } N$. Then M is locally weakly flat at each point of some neighborhood of $p \in \text{int } M$ if and only if M is locally flat at p.*

It should be noted that Theorem 3.3 together with results of Kirby [13] and Cantrell and Lacher [6] could be used to prove Theorem 4.2 without appealing to Theorem 4.1. However these results cannot be used in the case of manifolds with boundary in E^4 (see Theorem 5.1). Hence we are forced to appeal to the construction in Theorem 4.1 which gives a proof in all cases.

5. **Locally weakly flat on the boundary.** The definition of locally weakly flat may be extended to boundary points as follows. The manifold M is *locally weakly flat* at the point p of bd M if for each $\varepsilon > 0$ there is a neighborhood B of p in N of diameter less than ε such that B is an n-cell, $B \cap M = D$ is an m-cell, $D \cap \text{bd } B$ is an $(m-1)$-cell of bd D that is locally flat in bd B, $D \cap \text{bd } B$ has a collar in D compatible with a collar of B. The boundary version of Theorem 4.2 is as follows:

THEOREM 5.1. *Let M and N be m- and n-manifolds such that $M \subset \text{int } N$ and $m \leq n$. When $m < n$ suppose that M is locally weakly flat at each point of $U \cap \text{int } M$ where U is a neighborhood of $p \in \text{bd } M$. Then M is locally flat at p if and only if M is locally weakly flat in some neighborhood of p.*

Proof. In the case $n > 3$ the theorem is proved in much the same way that Theorem 4.2 was proved. Using Theorem 3.3 we construct an arc in bd M which contains the singular set of $M \cap U$ and which is tame in M and in N. Shrinking certain subarcs to a point we obtain for each $\varepsilon > 0$ a finite number of n-cells $B_1, \ldots, B_k$ such that $B_i \cap B_j = \varnothing$ for $i \neq j$, diam $B_i < \varepsilon$, $(B_i, B_i \cap M)$ is a semicell pair of type (n, m) which is nice on its boundary, and $Y \subset \bigcup_{i=1}^k \text{int } B_i$ where Y is the singular set of $M \cap U$. (The Lacher result [14] is used here.) Now by a procedure completely analogous to the proof of Theorem 4.1 we show using an appropriate form of Lemma 3.1 that M is locally flat at p.

If $n = 3$ and $m = 1$ then M is LPU at p. Since p is an isolated singular point, the theorem follows from well-known 3-space results. If $n = 3$ and $m = 2$ then bd M is LPU at p and clearly LU at p; therefore bd M is locally flat at p [11]. Hence the theorem follows from [15]. When $n = 3$ and $m = 3$ then bd M is LPU at each point of some neighborhood of p. Therefore bd M is locally flat at p [9]. This completes the proof of Theorem 5.1.

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